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The Important Thing Is not (Always) Winning  
but Taking Part: Funding Public Goods with  
Contests

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# The Important Thing Is not (Always) Winning but Taking Part: Funding Public Goods with Contests\*

Marco Faravelli<sup>†</sup>

## Abstract

This paper considers a public good game with incomplete information affected by extreme free-riding. We overcome this problem through the implementation of a contest in which several prizes can be awarded. For any possible distribution of wealth we identify the necessary and sufficient conditions for the equilibrium allocations to be interior for all players. At interior solutions, it is optimal for the social planner to set the last prize equal to zero, but otherwise the total expected welfare is independent of the distribution of the total prize sum among the prizes. We prove that private provision via a contest Pareto-dominates both public provision and private provision via a lottery.

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# 1 Introduction

This paper looks at contests as a way to overcome the free-riding problem. It is well known that the public good provision resulting from individual voluntary contributions is generally sub-optimal, because of the incentive to free-ride associated with positive externalities (see for examples Bergstrom et al., 1986; Andreoni, 1988). While fund-raising mechanisms based on tax rewards and penalties can be designed to solve this problem (e.g. Groves and Ledyard, 1977; Walker, 1981), they are not available to private organisations with no coercive power, such as charities or civic groups. Contests as incentive mechanisms are different from the above solutions because no power to enforce sanctions is required on the part of the institution conducting the tournament.

Morgan (2000) analyses lotteries, where contributions to the public good entitle to lottery tickets, as a means to solve the free-riding problem. He considers a model with quasilinear preferences where all players contribute the same amount. Morgan and Sefton (2000) provide experimental results in the case where all subjects have the same endowment, showing that contribution under a lottery is generally higher than under the voluntary contribution mechanism. Both these papers consider the case in which one prize is awarded. However, we will prove that a lottery with one prize performs worse than a contest, either with one or more prizes.

Contests are competitions in which agents spend resources in order to win one or more prizes. The main characteristic is that, independently of success, all participants bear some costs. There exists a large literature which analyses the use of contests and tournaments as incentive schemes<sup>1</sup>. However, they have only recently started to be examined as a means to overcome the free-riding problem and provide socially desirable public goods<sup>2</sup>.

The first price all-pay auction with complete information has been utilised extensively in the literature (Dasgupta, 1986; Hillman and Samet, 1987; Hillman and Riley, 1989; Ellingsen, 1991; Baye et al., 1993). There exists no pure strategy Nash equilibrium and a complete characterisation of its equilibria appears in Baye et al. (1996). Barut and Kovenock (1998) extend the analysis to the case of symmetric multiple prize all-pay auctions with complete information. They show that, when players are not constrained, only mixed strategy equilibria exist. Further, expected

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<sup>1</sup>Applications have been made to promotions in labour markets (Lazear and Rosen, 1981), technological and research races (Wright, 1983; Dasgupta, 1986; Taylor, 1995; Fullerton and McAfee, 1999; Windham, 1999), credit markets (Breocker, 1990), and rent seeking (Tullock, 1980) among others.

<sup>2</sup>In an independent study, Orzen (2005) analyses a first price all-pay auction as an incentive-based funding mechanism for the private provision of public goods. Orzen considers a linear public good game with complete information where all players have the same endowment. A prize is awarded to the individual who contributes the most. The author proves that a first price all-pay auction performs better than a lottery. However, he provides experimental results showing no statistical difference between the lottery and the auction.

expenditures are maximised by driving the value of the lowest prize to zero, but are invariant across all configurations leaving the lowest value fixed and the sum of the values constant.

In this paper we consider a linear public good game with heterogenous endowments which are private information. Such a game is a modified version of the game with complete information which is typically employed in public good experiments. Each agent chooses how much of her wealth to allocate to the public good; this money is multiplied by a parameter, which takes a value between one and the number of players, and shared equally among all the agents. The unique Nash equilibrium is to contribute nothing, although it is socially optimum to contribute all the wealth. We overcome this extreme free-riding via a contest where several prizes may be awarded. We assume that the social planner has access to a budget, which can be allocated in form of prizes. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution and so on until all prizes are awarded. The social planner wants to maximise the expected total welfare net of the value of the total prize sum.

The puzzle with the equilibrium defined in Morgan (2000) is that it predicts that agents with different endowments contribute the same amount, which does not seem realistic. In our model, at interior solutions, heterogeneity and incomplete information allow us to find a monotone equilibrium, in which the contribution is strictly increasing in the endowment. Such an equilibrium is a purification of the mixed strategy equilibrium defined by Barut and Kovenock (1998) and seems to be more plausible than a completely symmetric equilibrium, either in mixed or in pure strategy. For any possible distribution of wealth we identify the necessary and sufficient conditions for the equilibrium allocations to be interior for all players. As we said, we assume the typical utility function that is used in laboratory experiments on public goods. Although it does not seem plausible that in real life people spend all their wealth in auctions or lotteries, in the framework we analyse we believe that wealth constraints may be binding. We find that there exists a critical level of budget under which wealth constraints are non-binding for all agents. When the total prize sum is below such a critical value, it is optimal for the social planner to set the last prize equal to zero, but otherwise the total expected contribution is independent of the distribution of the total prize sum among the prizes. Further, provided interior solutions, we prove that private provision via a contest such as the one considered here Pareto-dominates both public provision and private provision via a lottery in which one prize is awarded.

In Section 2 we present a linear public good game with complete information. In Section 3 we present the model and identify the Nash equilibrium. In Section 4 we find necessary and sufficient conditions for interior solutions and we present the revenue equivalence result. Section 5 compares private provision via a contest with both public provision and private provision via a lottery. Section 6 concludes.

## 2 A Linear Public Good Game with Complete Information

In this section we present the game that is typically used in public good experiments  $n$  subjects take part in the experiment. Each subject is endowed with the same amount of money  $z$  and simultaneously chooses how much of her wealth to allocate to the public good; this money is multiplied by a parameter  $\alpha$  and shared equally among all the subjects. Agent  $i$ 's payoff can be described by

$$U_i = z - g_i + \alpha \frac{G}{n} \quad (1)$$

where  $g_i$  is  $i$ 's contribution to the public good and  $G = \sum_{i=1}^n g_i$  is the total level of public good. If  $\alpha \in (1, n)$  an individual opportunity cost of contributing to the public good exceeds the marginal return of investing in the public good. Thus, the unique Nash equilibrium of the game is to contribute nothing, while it is efficient to contribute  $z$ .

## 3 The Model

Let us consider  $n$  players. Each player  $i$  is assumed to have endowment  $z_i$ , which is private information. Endowments are drawn independently of each other from the interval  $[0, 1]$  according to the distribution function  $F(z)$ , which is common knowledge, with mean  $E[z]$ . We assume that  $F(z)$  has a continuous and bounded density  $F'(z) > 0$ . Players play a public good game in which each individual has to choose how much to contribute to the public good. At the same time they take part in a contest in which  $n$  prizes are awarded such that  $\pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0$ ,  $1 < m \leq n$  and  $\sum_{j=1}^n \pi_j = \Pi$ . This assumption rules out the possibility of awarding  $n$  equal prizes and will enable us find an equilibrium. We will call  $\pi = (\pi_1, \dots, \pi_n) \in \mathbb{R}^n$  the vector of prizes. The player with the highest contribution wins  $\pi_1$ , the player with the second highest contribution wins  $\pi_2$ , and so on until all the prizes are allocated. For each player, a strategy  $g(z)$  will be the contribution to the public good as a function of the player's endowment and the action space for player  $i$  will be the interval  $[0, z_i]$ . If player  $i$ , who has endowment  $z_i$  and contributes  $g_i$ , wins prize  $j$  her payoff is

$$U_i = z_i - g_i + \alpha \frac{G}{n} + \pi_j \quad (2)$$

where  $\alpha \in (1, n)$ .

Each player  $i$  chooses her contribution in order to maximise expected utility (given the other players' contributions and given the values of the different prizes). We will assume that  $\Pi$  is exogenously determined. For a given value of  $\Pi$ , the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximise the expected value of total welfare net of the value of  $\Pi$  (given the players' equilibrium strategy functions).

In this section we will focus on the case in which the equilibrium strategy  $g(z)$  is less than  $z$  for any type  $z$  on the interval  $[0, 1]$ .

In order to find the equilibrium of the game it is useful to present the function

$$K(F(z)) = \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \quad (3)$$

Given a vector of prizes  $\pi$ ,  $K(F(z))$  is a linear combination of  $n$  order statistics with weights equal to the prizes. If all agents adopt the same strictly increasing strategy  $g(z)$ ,  $K(F(z))$  represents the expected prize of the player with endowment  $z$ .

**Lemma 1** *The function  $K(F(z))$  is strictly monotonic increasing in  $z$ .*

**Proof.** Let's consider  $z_i$  and  $z_j$  such that  $0 \leq z_i < z_j \leq 1$ . Given that  $F(z_i) < F(z_j)$ , and given the assumption that  $\pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0$ ,  $1 < m \leq n$ ,  $K(F(z_j))$  assigns higher weights than  $K(F(z_i))$  to higher prizes and lower weights than  $K(F(z_i))$  to lower prizes. Therefore  $K(F(z_i)) < K(F(z_j))$ . ■

At interior solutions for all players, we are able to characterise a monotone equilibrium. Later on we will identify the necessary and sufficient conditions for its existence.

**Proposition 1** *Given a vector  $\pi$  of prizes, at an interior solution for all players the game has a symmetric pure strategy equilibrium given by*

$$g(z) = \frac{n}{n-\alpha} (K(F(z)) - \pi_n)$$

**Proof.** The expected utility of a player from a choice  $g$  can be calculated as

$$E[U(z-g, \pi) \mid g, g_{-i}] = z - g + \alpha \frac{G}{n} + (\Pr[1 \mid g, g_{-i}] \pi_1 + \Pr[2 \mid g, g_{-i}] \pi_2 + \dots + \Pr[n \mid g, g_{-i}] \pi_n)$$

where  $\Pr[j \mid g, g_{-i}]$  is the probability of a choice  $g$  being  $j$ -th highest conditional on the other strategies  $g_{-i}$ . If all agents adopt the same strictly increasing strategy  $g(z)$ , then the probability that a candidate with endowment  $z_i$  is higher ranked

than another randomly chosen candidate is  $\Pr[g(z_i) > g(z)] = \Pr[z_i > z] = F(z_i)$ . Therefore

$$\begin{aligned} & (\Pr[1 \mid g, g_{-i}] \pi_1 + \Pr[2 \mid g, g_{-i}] \pi_2 + \dots + \Pr[n \mid g, g_{-i}] \pi_n) = \\ K(F(z)) &= \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \end{aligned}$$

Now, given the common strategy  $g(z)$ , we suppose that an individual with endowment  $z$  chooses  $g(\hat{z})$  for some  $\hat{z}$ , then her expected utility will be

$$z - g(\hat{z}) + \alpha \frac{G_{-i} + g(\hat{z})}{n} + K(F(\hat{z}))$$

where  $G_{-i}$  is the sum of the contributions of all the other players. Differentiating with respect to  $\hat{z}$  we obtain

$$\frac{\alpha - n}{n} g'(\hat{z}) + K'(F(\hat{z})) F'(\hat{z})$$

In equilibrium the individual with endowment  $z$  should choose  $g(z)$  so that the above will be equal to zero when  $\hat{z} = z$ , and we have

$$g'(z) = \frac{n}{n - \alpha} K'(F(z)) F'(z)$$

A player with the lowest possible endowment  $z = 0$  does not contribute to the public good and wins the last prize. This yields the boundary condition  $g(0) = 0$ . Hence, the solution is

$$g(z) = \frac{n}{n - \alpha} (K(F(z)) - \pi_n)$$

From Lemma (1) we know that the candidate equilibrium function  $g$  is strictly monotonic increasing.

Assuming that all players rather than  $i$  play according to  $g$ , we finally need to show that, for any type  $z$  of player  $i$ , the contribution  $g(z)$  maximises the expected utility of that type. Let us consider an individual with endowment  $z$ . If she plays  $g(z) = \frac{n}{n-\alpha} (K(F(z)) - \pi_n)$  her expected utility is given by

$$E[U(z, g(z)) \mid g_{-i}] = z - \frac{\alpha}{n - \alpha} K(F(z)) + \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n} G$$

If she deviates and plays  $\frac{n}{n-\alpha} (K(F(\hat{z})) - \pi_n)$  for some  $\hat{z} \neq z$  her expected utility will be

$$\begin{aligned} E[U(z, g(\hat{z})) \mid g_{-i}] &= \\ & z - \frac{n}{n - \alpha} K(F(\hat{z})) + \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n} (G - \frac{n}{n - \alpha} K(F(z))) \\ & + \frac{n}{n - \alpha} \pi_n + \frac{n}{n - \alpha} K(F(\hat{z})) - \frac{n}{n - \alpha} \pi_n + K(F(\hat{z})) \\ & = z - \frac{\alpha}{n - \alpha} K(F(z)) + \frac{n}{n - \alpha} \pi_n + \frac{\alpha}{n} G \end{aligned}$$

Therefore she is indifferent to play any other strategy  $\frac{n}{n-\alpha}(K(F(\hat{z})) - \pi_n)$ . If her action space  $[0, z]$  is a subset of the set  $g_{-i}$  this rules out the possibility that she might be better off deviating from  $g(z)$ . If  $z > g(1)$  it is easy to show that she would be worse off playing any strategy greater than  $g(1)$ . In fact, playing  $g(1)$  would already guarantee  $\pi_1$  and any higher contribution would result in a lower expected utility. ■

## 4 Interior Solutions and Revenue Equivalence

In this section we are going to look for conditions that assure that the solution is interior for all players, given that the social planner wants to maximise the expected total welfare net of the value of the total prize sum. We will then analyse the expected total contribution when interior solutions are guaranteed.

First of all, let us present the social planner's maximisation problem, assuming that wealth constraints are non-binding for all players. Recall that  $\Pi$  is exogenously determined and the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximise the expected value of total welfare net of the value of  $\Pi$  (given the players' equilibrium strategy functions). This means that, to analyse the maximisation problem we have let the vector of prizes  $\pi$  be variable, maintaining the assumptions that  $\pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0$ ,  $1 < m \leq n$  and  $\sum_{j=1}^n \pi_j = \Pi$ , and we now have to study the family of functions

$$\begin{aligned} \phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \\ \dots = \pi_n \geq 0, 1 < m \leq n) = \\ \frac{n}{n-\alpha} \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \end{aligned} \quad (4)$$

Notice that, if  $\pi$  were fixed expression (4) would reduce to  $K(F(z))$ , as presented in (3).

Letting the vector of prizes  $\pi$  be variable, at an interior solution for all players,

the equilibrium strategy is represented by the following<sup>3</sup>

$$g(z, \pi) = \frac{n}{n - \alpha} (\phi(F(z), \pi) - \pi_n) = \frac{n}{n - \alpha} \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1 - F(z))^{i-1} - \frac{n}{n - \alpha} \pi_n$$

And the social planner's problem is given by

$$\max_{\pi} W = n \int_0^1 (z - g(z, \pi) + \frac{\alpha}{n} G + \phi(F(z), \pi)) F'(z) dz - \Pi \quad (5)$$

Notice first that

$$n \int_0^1 \phi(F(z), \pi) F'(z) dz = \Pi \quad (6)$$

independently of the distribution of the total prize sum among the different prizes.

Further, notice that, at interior solutions, we have

$$G = n \int_0^1 g(z, \pi) F'(z) dz = \frac{n}{n - \alpha} \Pi - \frac{n^2}{n - \alpha} \pi_n$$

This means that the expected total contribution only depends on the total prize sum and the value of the last prize.

Therefore, we can rearrange expression (5) as

$$\max_{\pi} W = n \int_0^1 (z - g(z, \pi) + \frac{\alpha}{n - \alpha} \Pi - \frac{\alpha n}{n - \alpha} \pi_n) F'(z) dz \quad (7)$$

and we can state the following result.

**Proposition 2** *At an interior solution for all players the social planner will set  $\pi_n = 0$ .*

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<sup>3</sup>For simplicity of notation, unless differently specified, from now on we will refer to  $\phi(F(z), \pi | \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_{m-1} > \pi_m = \dots = \pi_n \geq 0, 1 < m \leq n)$  as  $\phi(F(z), \pi)$ .

**Proof.** Expression (7) can be rewritten as

$$\begin{aligned} \max_{\pi} W &= n \int_0^1 \left( z - \frac{n}{n-\alpha} \phi(F(z), \pi) + \right. & (8) \\ &\quad \left. \frac{n}{n-\alpha} \pi_n + \frac{\alpha}{n-\alpha} \Pi - \right. \\ \left. \frac{\alpha n}{n-\alpha} \pi_n \right) F'(z) dz &= nE[z] + \frac{n(\alpha-1)}{n-\alpha} (\Pi - n\pi_n) \end{aligned}$$

It is obvious that  $\pi_n = 0$  maximises the above expression. ■

Provided that at an interior solution for all players the social planner will set the last prize equal to zero, Proposition (9) in Appendix A provides necessary and sufficient conditions for the value of  $\Pi$  such that  $g(z)$  is interior for any  $z$  on the interval  $[0, 1]$  for any possible distribution of  $\Pi$  among the first  $n - 1$  prizes. On the basis of this we can establish the following result.

**Proposition 3** *Provided that the last prize is equal to zero, there exists  $\bar{\Pi} > 0$  such that  $g(z)$  is interior for all players independently of the distribution of the total prize sum among the first  $n - 1$  prizes if and only if  $\Pi \leq \bar{\Pi}$ .*

**Proof.** See Appendix B. ■

As we noticed the expected total contribution only depends on the value of the total prize sum and on the last prize. We know, though, that if wealth constraints are non-binding for all players the social planner maximises the total expected welfare setting the last prize equal to zero. Therefore, at an interior solution the expected total contribution will be the same, independently of the distribution of the total prize sum among the first  $n - 1$  prizes. This result is summarised by the following proposition.

**Proposition 4** *If  $\Pi \leq \bar{\Pi}$  the social planner will set the last prize equal to zero and the expected total contribution will be  $G = \frac{n}{n-\alpha} \Pi$ , independently of the distribution of the total prize sum among the first  $n - 1$  prizes.*

Further, from the result above it is obvious that

**Corollary 1** *If  $\Pi \leq \bar{\Pi}$  the expected total contribution is strictly increasing in  $\Pi$ .*

## 5 Contest versus Public Provision and Lottery

In this section, we will compare the result obtained through a contest as the one we described with both the result generated by public provision and the one obtained using a lottery.

When socially desirable public goods are not privately provided the obvious alternative is to publicly provide them. Let us imagine that the social planner has access to a budget equal to  $\Pi \leq \bar{\Pi}$ . Instead of allocating this sum in form of prizes the social planner provides an amount of public good equal to  $\Pi$ . We want to compare the expected total welfare generated by such public provision with the expected total welfare resulting from the use of a contest, where the social planner awards up to  $n - 1$  prizes which sum is equal to  $\Pi$ .

**Proposition 5** *Private provision of public good via a contest, in which the total sum prize  $\Pi \leq \bar{\Pi}$  is distributed among the  $n - 1$  players who contribute the most, Pareto-dominates public provision. If the social planner uses  $\Pi \leq \bar{\Pi}$  to publicly provide the public good the expected total welfare net of the value of  $\Pi$  is  $W^P = nE[z] + (\alpha - 1)\Pi$ .*

**Proof.** If the social planner uses  $\Pi \leq \bar{\Pi}$  to provide the public good the expected total welfare net of the value of  $\Pi$  is given by

$$W^P = n \int_0^1 \left( z + \frac{\alpha}{n} \Pi \right) F'(z) dz - \Pi = nE[z] + (\alpha - 1)\Pi \quad (9)$$

From expression (8) we know that, if the last prize is equal to zero, the expected total welfare generated by a contest is equal to

$$W = nE[z] + \frac{n(\alpha - 1)}{n - \alpha} \Pi$$

that is strictly greater than (9). ■

We now consider the case where the social planner resorts to a lottery to encourage contribution to the public good. To be able to compare the use of a lottery with the use of a contest we will have to restrict the analysis to interior solutions. To do this let us assume  $n$  players whose endowments are drawn independently of each other from the interval  $[\underline{z}, \bar{z}]$ , with  $\underline{z}$  strictly positive, according to the distribution function  $F(z)$ , which is common knowledge. Assume that the social planner decides to award the sum  $\Pi$  via a lottery with the following properties. If a player  $i$  with endowment  $z_i$  contributes  $g_i \in [0, z_i]$  and the sum of the contributions of all other players is equal to  $G_{-i}$  she wins  $\Pi$  with probability  $\frac{g_i}{g_i + G_{-i}}$  and her expected utility is given by

$$E[U(z_i - g_i, \Pi) \mid g_i, G_{-i}] = z_i - g_i + \alpha \frac{G_{-i} + g_i}{n} + \frac{g_i}{g_i + G_{-i}} \Pi$$

Differentiating with respect to  $g$  and setting this equal to zero we have

$$\frac{\alpha - n}{n} + \frac{G_{-i}}{(g_i + G_{-i})^2} \Pi = 0$$

Assuming that the total contribution is different from zero<sup>4</sup> and rearranging we obtain player  $i$ 's best response function

$$g_i^* = -G_{-i} + \sqrt[2]{\frac{n}{n-\alpha}\Pi G_{-i}} \quad (10)$$

From (10) we can write an expression for the total contribution when player  $i$  plays according to her best response

$$G(g_i^* | G_{-i}) = \sqrt[2]{\frac{n}{n-\alpha}\Pi G_{-i}}$$

Although the endowment is private information, notice that  $z$  does not enter the first order condition. Each player will have the same best response function and at interior solutions the contribution in equilibrium will be the same for any  $z$ . Therefore, assuming interior solutions, we know that  $g_i^*$  will be  $\frac{G(g_i^* | G_{-i})}{n}$  and we can write

$$g_i^* = \frac{\sqrt[2]{\frac{n}{n-\alpha}\Pi G_{-i}}}{n} \quad (11)$$

Setting (10) and (11) equal we obtain an expression for  $G_{-i}$  when any player plays according to her best response function

$$G_{-i}^* = \frac{(n-1)^2}{n(n-\alpha)}\Pi$$

Hence we know that in equilibrium, at an interior solution for all players, all agents will play

$$g^* = \frac{n-1}{n(n-\alpha)}\Pi$$

And the total contribution in equilibrium, at an interior solution for all players, will be

$$G^* = \frac{n-1}{n-\alpha}\Pi$$

It is easy to see that if  $\Pi \leq \frac{n(n-\alpha)}{n-1}z$  the solution will be interior for all players. These results are summarised in the following proposition.

**Proposition 6** *Assume  $n$  players whose endowments are drawn independently of each other from the interval  $[z, \bar{z}]$ , with  $z$  strictly positive, according to the distribution function  $F(z)$ , which is common knowledge. Assume that  $z$  is private information. If  $\Pi \leq \frac{n(n-\alpha)}{n-1}z$  the lottery has a symmetric pure strategy equilibrium in which any player contributes  $g^* = \frac{n-1}{n(n-\alpha)}\Pi$  and the total contribution is  $G^* = \frac{n-1}{n-\alpha}\Pi$*

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<sup>4</sup>Notice that in equilibrium the total contribution will not be zero. In fact, if any other player different from  $i$  contributes zero, player  $i$  will contribute  $\varepsilon$  strictly positive and arbitrarily close to zero and win the prize.

It is interesting to notice that under such a lottery, unlike the contest, all players contribute the same amount in equilibrium.

Further, notice that in order to prove Proposition (8) in Appendix A we have not resorted to the support of  $z$  and that the same conditions guarantee that the solution will be interior for all players in the case in which endowments are drawn independently of each other from the interval  $[\underline{z}, \bar{z}]$ , with  $\underline{z}$  strictly positive, according to the distribution function  $F(z)$ , which is common knowledge, and which has a continuous and bounded density  $F'(z) > 0$ . Under a contest as the one described, provided that the social planner sets the last prize equal to zero, the total contribution is given by

$$\begin{aligned} G &= n \int_{\underline{z}}^{\bar{z}} \frac{n}{n-\alpha} \phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \\ &\dots \geq \pi_n, \pi_n = 0) F'(z) dz = \frac{n}{n-\alpha} \Pi \end{aligned}$$

Hence, we can conclude that, for  $\Pi$  that guarantees interior solutions for all players under both mechanisms, the expected total contribution raised with a contest is greater than the total contribution under a lottery, for any finite  $n$ .

**Proposition 7** *Assume  $n$  players whose endowments are drawn independently of each other from the interval  $[\underline{z}, \bar{z}]$ , with  $\underline{z}$  strictly positive, according to the distribution function  $F(z)$ , which is common knowledge. Assume that  $z$  is private information and that  $F(z)$  has a continuous and bounded density  $F'(z) > 0$ . If  $\Pi \leq \min[\frac{n(n-\alpha)}{n-1} \underline{z}, \bar{\Pi}]$ , the expected total provision of public good via a contest, in which the total sum prize is distributed among the  $n-1$  players who contribute the most, is greater than the total contribution raised under a lottery.*

## 6 Conclusions

Finding effective ways to fund public goods is an important policy question, given the role played by public goods in personal and collective well-being. There exists an extensive literature on fund-raising mechanisms based on taxes and penalties. However, solutions to the free-riding problem which do not require coercive power have only recently started to be analysed. In the case of institutions which are unable to enforce sanctions, such as charities, this difference may be of extreme importance.

In this paper we analysed the use of contests as incentive schemes to fund public goods. We considered a linear public good model as it is often employed in laboratory experiments. The main characteristics of the model are the possibility of awarding multiple prizes on the one side, and heterogeneity of the endowments and incomplete information on the other. We assumed that the social planner has access to a

budget and uses it to implement a contest. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution and so on until all prizes are awarded. The social planner's objective function is given by the expected total welfare net of the total sum prize. We found that there exists a critical level of budget under which wealth constraints are non-binding for all agents. For any possible distribution of wealth we identified the necessary and sufficient conditions for the equilibrium allocations to be interior for all players. When the total prize sum is below such a critical value it is optimal for the social planner to set the last prize equal to zero, but otherwise the total expected contribution is invariant to all configurations leaving the lowest value fixed. Provided interior solutions, we proved that a contest Pareto-dominates public provision of the public good and performs better than a lottery.

Heterogeneity of the endowments and incomplete information about income levels allowed us to characterise a monotone equilibrium. Such an equilibrium is a purification of the mixed strategy equilibrium described by Barut and Kovenock (1998). On the contrary, in the case of a lottery, a symmetric equilibrium arises (see Morgan, 2000). This is an interesting difference which makes the equilibrium of a contest looks more realistic than the latter. Indeed it does seem generally more plausible that richer people bid more than individuals with a lower income.

An interesting extension to the present work would be to test experimentally the main results of the model. First, an important question would be to check whether individuals actually contribute more in a contest with these characteristics than in lottery, and whether the revenue equivalence holds. Further, it would be interesting to test whether a monotone equilibrium would arise.

## Appendix A: Necessary and Sufficient Conditions

We want to find necessary and sufficient conditions for the value of  $\Pi$  such that  $g(z)$  is interior for any  $z$  on the interval  $[0, 1]$  for any possible allocation of  $\Pi$  among the first  $n - 1$  prizes. Notice in fact that, assuming interior solutions, Proposition (2) assures us that the social planner will set  $\pi_n = 0$ .

If we let the vector of prizes  $\pi$  be variable, provided that the last prize is equal to zero and that the sum of the first  $n - 1$  prizes is equal to  $\Pi$ ,  $g(z)$  is represented

by the following<sup>5</sup>

$$\begin{aligned} \frac{n}{n-\alpha} \phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_n, \pi_n = 0) = \\ \frac{n}{n-\alpha} \sum_{i=1}^n \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1} \end{aligned}$$

Let us define the following object.

**Definition 1** *Define the envelope function*

$$V(z) = \max_{\pi} \{ \phi(F(z), \pi) \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_n, \pi_n = 0 \}$$

for any  $z$  on the interval  $[0, 1]$ .

If we are able to provide necessary and sufficient conditions for  $V(z)$  to be weakly less than  $z$  for any  $z$  on the interval  $[0, 1]$ , it will be easy to extend the result to  $g(z)$ . In order to do this we will define some useful concepts that will help us in the course of our analysis.

**Definition 2** *For any  $i$  such that  $1 \leq i \leq n-1$ :*

- 1) *define the set  $Q^i \subset \mathbb{R}^n$  such that for every  $\pi \in Q^i$  it holds that  $\pi_1 \geq \dots \geq \pi_i > \pi_{i+1} = \dots = \pi_n = 0$  and  $\sum_{l=1}^i \pi_l = \Pi$ .*
- 2) *call  $\bar{\pi}^i$  the vector  $\pi \in Q^i$  such that  $\pi_1 = \dots = \pi_i = \frac{\Pi}{i}$ .*

**Definition 3** *For any  $i$  such that  $2 \leq i \leq n-1$  define the set  $\tilde{Q}^i \subset Q^i$  such that for every  $\pi \in \tilde{Q}^i$  it holds that  $\pi_1 > \pi_i$ .*

Obviously  $\bar{\pi}^1 \in Q^1$ , characterised by  $\pi_1^1 = \Pi, \pi_l^1 = 0$  for  $2 \leq l \leq n$ , is the only element of the set  $Q^1$  and  $\phi(F(z), \bar{\pi}^1) = \Pi(F(z))^{n-1}$ .

The next Proposition presents necessary and sufficient conditions for  $V(z)$  to be weakly less than  $z$  on the interval  $[0, 1]$ .

**Proposition 8**  *$\phi(F(z), \bar{\pi}^i) \leq z$  on the interval  $[0, 1]$  for  $1 \leq i \leq n-1$  are necessary and sufficient conditions for  $V(z) \leq z$ .*

---

<sup>5</sup>Notice that, unlike the rest of the paper, both in Appendix A and Appendix B, when writing  $\phi(F(z), \pi)$  we will refer to  $\phi(F(z), \pi \mid \sum_{j=1}^n \pi_j = \Pi, \pi_1 \geq \dots \geq \pi_n, \pi_n = 0)$ .

**Proof.** The necessity of these conditions is obvious. In order to prove sufficiency we will have to present some technical results.

**Lemma 2** Given a vector  $\pi^R \in \mathbb{R}^n$  such that  $\sum_{j=1}^n \pi_j^R = \Pi$  and  $\pi_1^R \geq \dots \geq \pi_n^R$ ,  $\pi_n^R = 0$ , consider a redistribution of the type  $-\Delta\pi_i^R = \Delta\pi_{i+1}^R$ , with  $1 \leq i \leq n-1$  and  $\Delta\pi_i^R > 0$ , and call the resulting vector  $\pi^S$ . Then,  $\phi(F(z), \pi^S) > \phi(F(z), \pi^R)$  for any  $z$  such that  $F(z) < \frac{n-i}{n}$  and  $\phi(F(z), \pi^S) < \phi(F(z), \pi^R)$  for any  $z$  such that  $F(z) > \frac{n-i}{n}$ .

**Proof.** Notice that  $\frac{\partial\phi(F(z), \pi)}{\partial\pi_i} = \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}$ . To see how a redistribution of the type  $-\Delta\pi_i = \Delta\pi_{i+1}$  affects  $\phi(F(z), \pi)$  we have to study the sign of

$$\begin{aligned} & -\frac{\partial\phi(F(z), \pi)}{\partial\pi_i} + \frac{\partial\phi(F(z), \pi)}{\partial\pi_{i+1}} \\ &= (F(z))^{n-i}(1-F(z))^{i-1} \left( -\binom{n-1}{i-1}(F(z)) \right. \\ & \quad \left. + \binom{n-1}{i}(1-F(z)) \right) \end{aligned} \tag{12}$$

It is the case that expression (12)  $> 0$  for any  $z$  such that  $F(z) < \frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}}$  and (12)  $< 0$  for any  $z$  such that  $F(z) > \frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}}$ . Further, it is easy to show that

$$\frac{\binom{n-1}{i}}{\binom{n-1}{i-1} + \binom{n-1}{i}} = \frac{\frac{(n-1)!}{i!(n-1-i)!}}{\frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-1)!}{i!(n-1-i)!}} = \frac{n-i}{n}$$

■

**Lemma 3** Assume  $1 \leq i \leq n-2$ . Consider a vector  $\pi^B \in \tilde{Q}^{i+1}$ . If  $2 \leq i \leq n-2$  then  $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$ . If  $i = 1$  then  $\phi(F(z), \bar{\pi}^2) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) < \frac{n-1}{n}$  and  $\phi(\frac{n-1}{n}, \pi \mid \pi \in Q^2) = \phi(\frac{n-1}{n}, \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$ .

**Proof.** Let us first consider the case in which  $2 \leq i \leq n-2$ . The vector  $\bar{\pi}^{i+1}$  can be obtained from vector  $\pi^B$  applying the following algorithm in  $i$  steps.

**Algorithm 1** Step 1. From vector  $\pi^B$  construct vector  $\pi^{B1}$  such that  $\pi_1^{B1} = \frac{\Pi}{i+1}$ ,  $\pi_2^{B1} = \pi_2^B + \pi_1^B - \frac{\Pi}{i+1}$ ,  $\pi_j^{B1} = \pi_j^B$ ,  $3 \leq j \leq i+1$ . Given that  $\pi_2^B \geq \pi_3^B$  it will now be the case that  $\pi_2^{B1} > \pi_3^{B1} \geq \dots \geq \pi_{i+1}^{B1}$ . Therefore  $\frac{\Pi}{i+1} + i\pi_2^{B1} > \Pi$ . The last inequality can be rewritten as  $\pi_2^{B1} > \frac{\Pi}{i+1}$ , therefore we can move to the next step and repeat the process.

Step  $j$ , with  $2 \leq j \leq i - 1$ . From vector  $\pi^{Bj-1}$  construct vector  $\pi^{Bj}$  such that  $\pi_j^{Bj} = \frac{\Pi}{i+1}$ ,  $\pi_{j+1}^{Bj} = \pi_{j+1}^{Bj-1} + \pi_j^{Bj-1} - \frac{\Pi}{i+1}$ ,  $\pi_l^{Bj} = \pi_l^{Bj-1}$  for  $1 \leq l \leq j - 1$  and  $j + 1 \leq l \leq i + 1$ . Given that  $\pi_{j+1}^{Bj-1} \geq \pi_{j+2}^{Bj-1}$  it will now be the case that  $\pi_{j+1}^{Bj} > \pi_{j+2}^{Bj} \geq \dots \geq \pi_{i+1}^{Bj}$ . Therefore it is the case that  $2\frac{\Pi}{i+1} + (i + 1 - j) \pi_{j+1}^{Bj}$ . Rearranging the last inequality we obtain  $\pi_{j+1}^{Bj} > \frac{\Pi}{i+1}$ . This means that we can move to the next step and repeat the process.

Step  $i$ . From vector  $\pi^{Bi-1}$  construct vector  $\pi^{Bi}$  such that  $\pi_i^{Bi} = \frac{\Pi}{i+1}$ ,  $\pi_{i+1}^{Bi} = \pi_{i+1}^{Bi-1} + \pi_i^{Bi-1} - \frac{\Pi}{i+1}$ ,  $\pi_l^{Bi} = \pi_l^{Bi-1}$  for  $1 \leq l \leq i - 1$ . Notice that  $\pi_l^{Bi-1} = \frac{\Pi}{i+1}$  for  $1 \leq l \leq i - 1$ . Therefore  $\pi^{Bi} = \bar{\pi}^{i+1}$ .

Notice that from Lemma 2 we know that  $\phi(F(z), \pi^{Bj}) > \phi(F(z), \pi^{Bj-1})$  for any  $z$  such that  $F(z) < \frac{n-j}{n}$  for  $1 \leq j \leq i$ . Therefore  $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$ , that contradicts our assumption.

Consider now the case in which  $i = 1$ . Notice that  $\bar{\pi}_1^2 < \pi_1^B$  and  $\bar{\pi}_2^2 > \pi_2^B$ . Applying the same algorithm as above from  $\pi^B$  we will obtain  $\bar{\pi}_2^2$  after the first step. Applying Lemma 2 we know that  $\phi(F(z), \bar{\pi}^2) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) < \frac{n-1}{n}$ . Further, from Lemma 2 we also know that  $\phi(F(z), \pi \mid \pi \in Q^2) > \phi(F(z), \bar{\pi}^1)$  for any  $z$  such that  $F(z) < \frac{n-1}{n}$  and  $\phi(F(z), \pi \mid \pi \in Q^2) < \phi(F(z), \bar{\pi}^1)$  for any  $z$  such that  $F(z) > \frac{n-1}{n}$ . Therefore, by continuity, we can conclude that  $\phi(\frac{n-1}{n}, \pi \mid \pi \in Q^2) = \phi(\frac{n-1}{n}, \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$ . ■

**Lemma 4** Assume  $2 \leq i \leq n - 2$ .  $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in Q^j)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$  and for  $1 \leq j \leq i$ .

**Proof.** The structure of this proof is in three parts.

First of all, from Lemma 3 we know that  $\phi(F(z), \bar{\pi}^j) > \phi(F(z), \pi \mid \pi \in \tilde{Q}^j)$  for any  $z$  such that  $F(z) \leq \frac{n-j-1}{n}$  and, given that  $1 \leq j \leq i$ , for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$ .

For the second part of the proof, let us first assume  $j = 1$ . Consider a vector  $\pi^B \in \tilde{Q}^{i+1}$ . We want to show that  $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^1)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$ .

If  $2 \leq j \leq i$ , consider a vector  $\pi^B \in \tilde{Q}^{i+1}$  such that  $\pi_l^B = \bar{\pi}_l^j$  for  $1 \leq l \leq j - 1$ . Notice that, obviously,  $\pi_j^B < \bar{\pi}_j^j$ . We want to show that

$\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$  if  $1 \leq j \leq i - 1$  and for any  $z$  such that  $F(z) < \frac{n-i}{n}$  if  $j = i$ .

Vector  $\pi^B$  can be obtained from  $\bar{\pi}^j$  through the following algorithm in  $i + 1 - j$  steps.

**Algorithm 2** Step 1. If  $j = 1$ , from vector  $\bar{\pi}^1$  construct vector  $\pi^{A1} \in \tilde{Q}^2$  such that  $\pi_1^{A1} = \pi_1^B$  and  $\pi_2^{A1} = \Pi - \pi_1^B$ . If  $2 \leq j \leq i$ , from vector  $\bar{\pi}^j$  construct vector  $\pi^{A1} \in \tilde{Q}^{j+1}$  such that  $\pi_l^{A1} = \bar{\pi}_l^j = \frac{\Pi}{j}$  for  $1 \leq l \leq j - 1$ ,  $\pi_j^{A1} = \pi_j^B$  and  $\pi_{j+1}^{A1} = \bar{\pi}_j^j - \pi_j^B = \frac{\Pi}{j} - \pi_j^B$ .

Step  $k$ , with  $2 \leq k \leq i - j$ . From vector  $\pi^{Ak-1}$  construct vector  $\pi^{Ak} \in \tilde{Q}^{j+k}$  such that  $\pi_l^{Ak} = \pi_l^{Ak-1}$  for  $1 \leq l \leq j = k - 2$ ,  $\pi_{j+k-1}^{Ak} = \pi_{j+k-1}^B$  and  $\pi_{j+k}^{Ak} = \pi_{j+k-1}^{Ak-1} - \pi_{j+k-1}^B$ .

Step  $i + 1 - j$ . From vector  $\pi^{Ai-j}$  construct vector  $\pi^{Ai+1-j} \in \tilde{Q}^{i+1}$  such that  $\pi_l^{Ai+1-j} = \pi_l^{Ai-j}$  for  $1 \leq l \leq i - 2$ ,  $\pi_i^{Ai+1-j} = \pi_i^B$  and  $\pi_{i+1}^{Ai+1-j} = \pi_i^{Ai-j} - \pi_i^B$ . Notice that  $\pi_{i+1}^{Ai+1-j} = \pi_{i+1}^B$  and  $\pi^{Ai+1-j} = \pi^B$  by construction.

From Lemma 2 we know that  $\phi(F(z), \pi^{Ak}) > \phi(F(z), \pi^{Ak-1})$  for any  $z$  such that  $F(z) < \frac{n+1-k}{n}$ . Therefore if  $1 \leq j \leq i - 1$  then  $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$ . If  $j = i$  then  $\phi(F(z), \pi^B) > \phi(F(z), \bar{\pi}^j)$  for any  $z$  such that  $F(z) < \frac{n-i}{n}$  and  $\phi(\frac{n-i}{n}, \pi^B) = \phi(\frac{n-i}{n}, \bar{\pi}^j)$ .

Finally, from Lemma 3 we know that  $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$ . Therefore  $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in \tilde{Q}^j)$  for any  $z$  such that  $F(z) \leq \frac{n-1}{n}$ . ■

**Lemma 5** Assume  $2 \leq i \leq n - 2$ . Consider a vector  $\pi^B \in \tilde{Q}^{i+1}$  such that  $\pi_1^B > \pi_j^B$ , with  $2 \leq j \leq i$ . Assume a vector  $\pi^C \in \tilde{Q}^{i+1}$  such that  $\pi_l^C = \pi_l^B$  for  $j + 1 \leq l \leq i + 1$

and  $\pi_1^C = \dots = \pi_j^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ . If  $3 \leq j \leq n - 2$  then  $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) \leq \frac{n-j+1}{n}$ . If  $j = 2$  then  $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) < \frac{n-1}{n}$  and  $\phi(\frac{n-1}{n}, \pi^C) > \phi(\frac{n-1}{n}, \pi^B)$ .

**Proof.** Notice that  $\pi_1^B > \pi_1^C$  and  $\pi_j^B < \pi_j^C$ . Vector  $\pi^C$  can be obtained from vector  $\pi^B$  applying the following algorithm in  $j - 1$  steps.

**Algorithm 3** Step 1. From vector  $\pi^B$  construct vector  $\pi^{B1}$  such that  $\pi_1^{B1} = \pi_1^C$ ,  $\pi_2^{B1} = \pi_2^B + \pi_1^B - \pi_1^C$ ,  $\pi_l^{B1} = \pi_l^B$  for  $3 \leq l \leq i + 1$ . Given that  $\pi_2^B \geq \pi_3^B$  it will now be the

case that  $\pi_2^{B1} > \pi_3^{B1} \geq \dots \geq \pi_{i+1}^{B1}$ . Therefore  $\pi_1^{B1} + (j - 1)\pi_2^{B1} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$ . Since

$\pi_1^{B1} = \pi_1^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ , the last inequality can be rearranged as  $\pi_2^{B1} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ .

Therefore we can move to the next step and repeat the process.

Step  $k$ ,  $2 \leq k \leq j - 2$ . From vector  $\pi^{Bk-1}$  construct vector  $\pi^{Bk}$  such that  $\pi_k^{Bk} = \pi_k^C$ ,  $\pi_{k+1}^{Bk} = \pi_{k+1}^{Bk-1} + \pi_k^{Bk-1} - \pi_k^C$ ,  $\pi_l^{Bk} = \pi_l^{Bk-1}$  for  $1 \leq l \leq k - 1$  and

$k + 2 \leq l \leq i + 1$ . Notice that, by construction  $\pi_l^{Bk} = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$  for  $1 \leq l \leq k$  and  $\pi_l^{Bk} = \pi_l^B$  for  $k + 2 \leq l \leq i + 1$ . Given that  $\pi_{k+1}^{Bk-1} \geq \pi_{k+2}^{Bk-1}$  it will now be the case that  $\pi_{k+1}^{Bk} > \pi_{k+1}^{Bk} \geq \dots \geq \pi_{i+1}^{Bk}$ . Therefore  $\frac{k}{j}(\Pi - \sum_{l=j+1}^{i+1} \pi_l^B) + (j - k)\pi_{k+1}^{Bk} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$ .

$$\Pi - \sum_{l=j+1}^{i+1} \pi_l^B$$

The last inequality can be rearranged as  $\pi_{k+1}^{Bk} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ . Therefore we can move to the next step and repeat the process.

*Step  $j-1$ .* From vector  $\pi^{Bj-2}$  construct vector  $\pi^{Bj-1}$  such that  $\pi_{j-1}^{Bj-1} = \pi_{j-1}^C$ ,  $\pi_j^{Bj-1} = \pi_j^{Bj-2} + \pi_{j-1}^{Bj-2} - \pi_{j-1}^C$ ,  $\pi_l^{Bj-1} = \pi_l^{Bj-2}$  for  $1 \leq l \leq j-2$  and  $j+1 \leq l \leq i+1$ . Notice that  $\pi^{Bj-1} = \pi^C$  by construction.

From Lemma 2 we know that  $\phi(F(z), \pi^{Bk}) > \phi(F(z), \pi^{Bk-1})$  for any  $z$  such that  $F(z) < \frac{n-k}{n}$  for  $1 \leq k \leq j-1$ . This means that if  $3 \leq i \leq n-3$  then, by construction, we will have  $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) \leq \frac{n-j+1}{n}$ . If  $i = 2$  then  $j$  will necessarily be equal to 3 and, by construction, we will have  $\phi(F(z), \pi^C) > \phi(F(z), \pi^B)$  for any  $z$  such that  $F(z) < \frac{n-1}{n}$ . Further it will be the case that  $\phi(\frac{n-1}{n}, \pi^C) > \phi(\frac{n-1}{n}, \pi^B)$ . ■

**Lemma 6** Consider a vector  $\pi^C \in \tilde{Q}^{i+1}$  such that  $\pi_{i+1}^C = x$ ,  $\pi_j^C = \frac{\Pi-x}{i}$  with  $0 < x < \frac{\Pi}{i+1}$  for  $1 \leq j \leq i$  and  $2 \leq i \leq n-2$ . If  $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$  then  $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$ .

**Proof.** The inequality  $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$  can be rewritten as

$$\begin{aligned} & \frac{\Pi-x}{i} ((F(z))^{n-1} + \dots + \\ & \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1}) + \\ & x \binom{n-i}{i} (F(z))^{n-i-1} (1-F(z))^i - \\ & \frac{\Pi}{i+1} ((F(z))^{n-1} + \dots + (F(z))^{n-i-1} (1-F(z))^i) > 0 \end{aligned}$$

The above expression can be rearranged as

$$\begin{aligned} & \left( \frac{\Pi-x}{i} - \frac{\Pi}{i+1} \right) ((F(z))^{n-1} + \dots + \\ & \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1}) > \\ & \left( \frac{\Pi}{i+1} - x \right) (F(z))^{n-i-1} (1-F(z))^i \end{aligned} \quad (13)$$

Call  $A$  the expression  $((F(z))^{n-1} + \dots + \binom{n-1}{i-1} (F(z))^{n-i} (1-F(z))^{i-1})$  and call  $B$  the expression  $(F(z))^{n-i-1} (1-F(z))^i$ . Inequality (13) is satisfied for  $\frac{A}{B} > i$ .

The inequality  $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$  can be rewritten as

$$\frac{\Pi}{i} A - \frac{\Pi-x}{i} A - xB > 0 \quad (14)$$

Inequality (14) is satisfied for  $\frac{A}{B} > i$ . ■

From Lemma (4) we know that  $\phi(F(z), \bar{\pi}^{i+1}) > \phi(F(z), \pi \mid \pi \in Q^j)$  for any  $z$  such that  $F(z) \leq \frac{n-i}{n}$  and for  $2 \leq i \leq n-2$  and  $1 \leq j \leq i$ . In particular, this means that  $V(z)$  will be equal to  $\phi(F(z), \bar{\pi}^{n-1})$  for any  $z$  such that  $0 \leq F(z) \leq \frac{2}{n}$ . For those  $z$  such that  $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$  we will have to check the family of functions  $\phi(F(z), \pi \mid \pi \in Q^{n-1})$  and  $\phi(F(z), \bar{\pi}^{n-2})$ . In general, assuming  $0 \leq i \leq n-3$ , in order to find  $V(z)$  for those  $z$  such that  $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$  we will have to check the families of functions  $\phi(F(z), \pi \mid \pi \in Q^j)$  for  $i+2 \leq j \leq n-1$  and the function  $\phi(F(z), \bar{\pi}^{i+1})$ .

Consider now a vector  $\pi^C \in Q^{i+1}$  such that  $\pi_1^C = \dots = \pi_i^C > \pi_{i+1}^C$ , for  $2 \leq i \leq n-2$ . From Lemma (5) we know that, for those  $z$  such that  $\frac{n-i}{n} < F(z) \leq \frac{n-i+1}{n}$ , the function  $\phi(F(z), \pi^C)$  is greater than any other function of the family  $\phi(F(z), \pi \mid \pi \in Q^{i+1})$  with exclusion of  $\phi(F(z), \bar{\pi}^{i+1})$ .

From Lemma (6) though, we know that if  $\phi(F(z), \pi^C) > \phi(F(z), \bar{\pi}^{i+1})$  then it is the case that  $\phi(F(z), \bar{\pi}^i) > \phi(F(z), \pi^C)$ .

Therefore, in order to find the envelope function  $V(z)$  for those  $z$  such that  $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$ , it will be sufficient to check the two functions  $\phi(F(z), \bar{\pi}^{n-1})$  and  $\phi(F(z), \bar{\pi}^{n-2})$ . In general, assuming  $0 \leq i \leq n-3$ , in order to find  $V(z)$  for those  $z$  such that  $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$  we will have to check the functions  $\phi(F(z), \bar{\pi}^j)$  for  $i+1 \leq j \leq n-1$ .

From this follows that  $\phi(F(z), \bar{\pi}^i) \leq z$  for  $1 \leq i \leq n-1$  are sufficient conditions for  $V(z) \leq z$  on the interval  $[0, 1]$ . ■

Finally, given Proposition (8), by continuity we can establish the following result.

**Proposition 9** *Provided that the last prize is equal to zero,  $g(z)$  is interior for any  $z$  on the interval  $[0, 1]$  independently of the distribution of  $\Pi$  among the first  $n-1$  prizes if and only if  $\frac{n}{n-\alpha} \phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \Pi) \leq z$  on the interval  $[0, 1]$  for  $1 \leq i \leq n-1$ .*

## Appendix B

**Proof of Proposition (3).** Given a vector  $\pi^A$  such that  $\pi_1^A \geq \dots \geq \pi_n^A$ ,  $\pi_n^A = 0$  and  $\sum_{j=1}^n \pi_j^A = \Pi^A$  consider  $\pi^B$  such that  $\pi_i^B = c\pi_i^A$  for any  $1 \leq i \leq n$ ,  $c > 1$ . Notice that  $\sum_{j=1}^n \pi_j^B = \Pi^B = c\Pi^A$ . Since  $\phi(F(z), \pi^B) = c\phi(F(z), \pi^A)$  we conclude that  $\phi(F(z), \pi^B) > \phi(F(z), \pi^A)$  for any  $z \in (0, 1]$ .

Consider  $\Pi$  such that  $\frac{n}{n-\alpha} \phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \Pi) > z$  for  $z \in (a, b)$  with  $0 \leq a < b \leq 1$  for  $i$  such that  $1 \leq i \leq n-1$ . Given that  $F(z)$  has a continuous

and bounded density there exists  $c > 1$  such that  $\frac{n}{n-\alpha}\phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \frac{\Pi}{c}) \leq z$  for any  $z \in [0, 1]$  and  $i$  such that  $1 \leq i \leq n - 1$ . Therefore, by continuity there exists  $\bar{\Pi} > 0$  such that  $\frac{n}{n-\alpha}\phi(F(z), \bar{\pi}^i \mid \sum_{j=1}^n \bar{\pi}_j^i = \Pi) \leq z$  on the interval  $[0, 1]$  for  $1 \leq i \leq n - 1$  if and only if  $\Pi \leq \bar{\Pi}$ .

Given the result presented in Proposition (8) we can conclude that, provided that the last prize is equal to zero,  $\Pi \leq \bar{\Pi}$  is a necessary and sufficient condition for the solution to be interior for all players independently of the distribution of  $\Pi$  among the first  $n - 1$  prizes. ■

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